Lesson 21 - First-Order Linear Differential Equations D.E.'s

I. Definitions and Reminders
II Finding general solutions to 1st-order linear D.E.'s in standard form.

Announcement

Take-home quiz 6 Due @ start of class on Mon 10/23

Definitions/Reminders

① A D.E. is a 1st-order linear D.E. if it can be written in the form

$$A(x)\frac{dy}{dx} + B(x)y + C(x) = 0$$

$Ax+By+C=0$ line

Keys: ① y only appears in $\frac{dy}{dx}$ & y No y^2, other ops on y.
② Everything else only involves x

The standard form for a 1st-order linear D.E. is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

TODAY - Solve 1st-order linear D.E.'s that are already in standard form.

$\boxed{\text{Ex}}$ Separable? 1st-order linear? Both? Neither?

(A) $\frac{dy}{dx} + x^2 y = 2x^2$ linear

also separable

$$\frac{dy}{dx} = 2x^2 - x^2 y$$

$$\frac{dy}{dx} = x^2(2-y)$$

(B) $\cos(x)\frac{dy}{dx} + 2xy = x^3 + 2$ linear only

$\cos(x)\frac{dy}{dx} + 2xy - (x^3+2) = 0$

$$\frac{dy}{dx} = \underbrace{\frac{x^3+2-2xy}{\cos(x)}}$$

NOT separable

can't write RHS as

$f(x)\,g(y)$

(C) $\frac{dy}{dx} = xy^2$ separable

NOT 1st-order linear

because of square (^{2})

on y.

Reminders

① $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$

② $\int \frac{d}{dx}\left[f(x)\right]dx = f(x) + C$

③ $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$

④ $\frac{d}{dx}\left[e^{\int f(x)dx}\right] = e^{\int f(x)dx} \cdot \frac{d}{dx}\left[\int f(x)d\right] = e^{\int f(x)dx} f(x)$ chain rule

Std form: $\frac{dy}{dx} + P(x)y = Q(x)$

II. Finding sol'ns of 1st-order linear D.E.'s in standard form:

$\boxed{\text{Ex}}$

$$\frac{dy}{dx} + 3y = 5$$

1st order linear
std form: $P(x) = 3$
$Q(x) = 5$

① We are going to make LHS look like it came from product rule.

Rabbit: Multiply both sides by e^{3x}

integrating factor.

$$\underbrace{e^{3x}\frac{dy}{dx} + 3e^{3x}y}_{\text{Do you see product rule?}} = 5e^{3x}$$

$$\frac{d}{dx}\left[e^{3x}y\right]$$

$$\frac{d}{dx}\left[e^{3x}y\right] = 5e^{3x}$$

$$\int \frac{d}{dx}\left[e^{3x}y\right]dx = \int 5e^{3x}dx$$

$u = 3x$

$$\rightarrow \boxed{e^{3x}}\boxed{y} = \frac{5e^{3x}}{3} + C$$

integrating factor

$\int$ integrating factor $\cdot Q(x)dx$

$$y = \frac{5/3\, e^{3x} + C}{e^{3x}}$$

$$y = \frac{5}{3} + \frac{C}{e^{3x}} \quad \text{OR} \quad y = \frac{5}{3} + Ce^{-3x}$$

Procedure for solving 1st-order linear D.E.'s in std form

$$\frac{dy}{dx} + P(x)\, y = Q(x) \qquad (*)$$

① Calculate the integrating factor

$$\beta(x) = e^{\int P(x)dx}$$

Don't add constant to $\int P(x)\, dx$

Last ex: $P(x) = 3$

$$\beta(x) = e^{\int 3\, dx} = e^{3x}$$

② Multiply both sides of $(*)$ by $\beta(x)$

$$\underbrace{e^{\int P(x)dx}}_{1st} \underbrace{\frac{dy}{dx}}_{2nd'} + \underbrace{e^{\int P(x)dx} P(x)}_{1st'} \underbrace{y}_{2nd} = e^{\int P(x)dx} Q(x)$$

③ Integrate both sides to find sol'n

$$e^{\int P(x)dx}\, y = \int e^{\int P(x)dx} Q(x)\, dx$$

$$\boxed{\beta(x)\, y = \int \beta(x)\, Q(x)\, dx}$$

+C when you do $\int$ on RHS

④ Divide by $\beta(x)$ to solve for y.

$\boxed{\text{Ex}}$ (Edwards, et.al §1.5 #4)

$$\frac{dy}{dx} - 2xy = e^{x^2}$$

1st-order linear $\quad P(x) = -2x$

std form ✓ $\quad Q(x) = e^{x^2}$

① $\rho(x) = e^{\int -2x\,dx} = e^{-x^2}$

③ $$ye^{-x^2} = \int \underbrace{e^{-x^2}\,e^{x^2}}_{e^{-x^2+x^2}}\,dx \qquad = e^0 = 1$$

$$ye^{-x^2} = \int 1\,dx$$

$$ye^{-x^2} = x + C$$

④ $$y = \frac{x+C}{e^{-x^2}} = \frac{x}{e^{-x^2}} + \frac{C}{e^{-x^2}}$$

$$y = xe^{x^2} + Ce^{x^2}$$

$\boxed{\text{Ex}}$ $$\frac{dy}{dx} + \overset{\frac{2y}{x}\ \checkmark}{\frac{2}{x}y} = x^4$$

Ans: $$y = \frac{x^5}{7} + \frac{C}{x^2}$$